Some exact solutions methods for the nonlinear Schrödinger equation

Niamh Newman
19341103
n.newman1@nuigalway.ie

Supervisor: Martin Meere

Table of Contents

1 Introduction and Background

1.1 Background

The nonlinear Schrödinger equation (NLSE) is a nonlinear partial differential equation that fascinates physicists and applied mathematicians because of its applications to many different nonlinear phenomena in fluid mechanics,[1] condensed matter physics,[2] and nonlinear optics.[3] In particular, in nonlinear optics the NLSE models nonlinear occurrences such as self-phase modulation, four-wave mixing and second harmonic generation, to name a few.[4] A good understanding of this equation's solutions and its properties may contribute to finding useful applications to these types of models. Since the principle of superposition does not apply to nonlinear differential equations, mathematicians have to look at each equation on a case by case basis and it is not always possible to derive an exact solution. However the NLSE is of particular interest because it can support soliton solutions. These are exact solutions which were discovered to solve certain nonlinear equations in the 1800s.[5]

Solitons are nonlinear stable waves that arise in media where the effects of dispersion and nonlinearity cancel out. Solitary waves were first observed by John Scott Russel in 1834.[6] He was a Scottish engineer who noticed waves that travelled from a boat did not dissolve into ripples as expected but remained stable and travelled many kilometres. He created his own solitary waves in a water tank and made observations. He discovered that these water waves would travel over long distances while the speed and shape remained stable. He observed that the speed of the wave depends on the depth of the water and that the waves never merge i.e. two solitary waves emerge after a collision retaining their shape and speed.

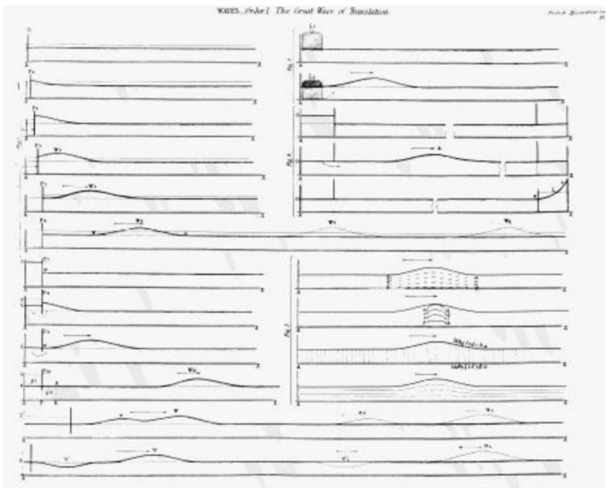

Figure 1: Russell's Observations of a Soliton[6]

This discovery remained controversial as they disagreed with Bernoulli's and Newton's theorems of hydrodynamics. The controversy ended when N. J. Zabusky and M. D. Kruskal demonstrated that the Korteweg-de Vries equation, another nonlinear partial differential equation, can be solved with a soliton.[5] Since then there has been growing interest in soliton solutions and their application in nonlinear phenomena.

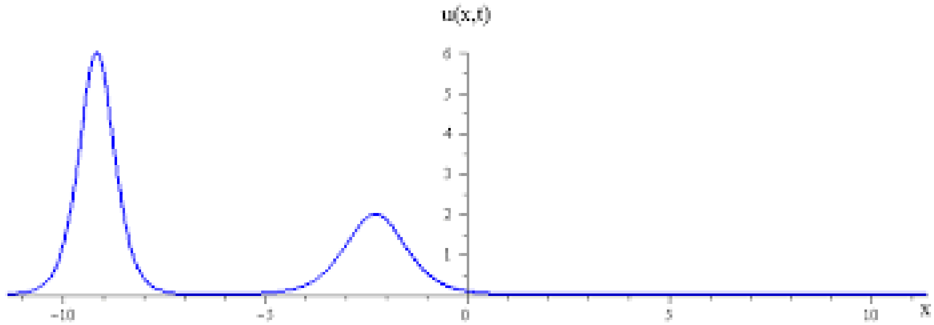

Figure 2: Graph of two solitons

1.2 Aims and objectives

In this paper soliton solutions will be derived from the cubic NLSE using two methods. The cubic NLSE takes the following form:

$$iu_t + au_{xx} + c|u|^{2m}u = 0 \tag{1}$$

where $u = u(x,t)$ is the complex-valued function of two real variables x, t, $i = \sqrt{-1}$ and the real coefficients, a and c, correspond to the group velocity and the nonlinear coefficient respectfully. $m > 0$ is the nonlinear parameter. For $m = 1$, $a = 1$ and $c = \mu$, the equation represents the NLSE. Equation (1) is similar to the equation that models light propagating through left-handed metamaterials. These are artificially-made materials that can affect EM waves differently compared to EM waves propagating through naturally occurring materials.[7]

There are many ways to solve this nonlinear equation such as the sine-cosine method,[8] the $(\frac{G'}{G})$-expansion method,[9] and the $\exp(-\phi(\xi))$ - expansion method[10] In this paper the solutions will be derived by the direct integration method and the simple equation method[11] under the travelling-wave assumption.

2 Direct Integration

We assume (1) is solved by seeking a travelling wave solution of the form:

$$u(x,t) = e^{i\theta(x,t)}U(\xi) \tag{2}$$

where $U(\xi)$ is a real function for the amplitude of the wave and $\xi = x - vt$, where v is real and corresponds to the speed of the wave. The phase component of the wave is $\theta(x,t) = -kx + \omega t + \theta_0$, where k is the wavenumber, ω is the angular frequency and θ_0 is the phase constant. All these quantities are real. To change the variables from (x,t) to ξ we use the relation $\xi = x - vt$ and the chain rule. Substituting (2) into (1), equation (1) becomes:

$$aU'' - 2iakU' - \omega U - ivU' - ak^2U + cU^{2m+1} = 0 \tag{3}$$

.
After splitting into real and imaginary parts, we obtain

$$(v + 2ak)U' = 0 \tag{4}$$

and

$$aU'' - (\omega + ak^2)U + cU^{2m+1} = 0 \tag{5}$$

.
The speed of the soliton is found from (4):

$$v = -2ak \tag{6}$$

.
Multiplying equation (5) by U' and integrating gives

$$(U')^2 - \left(\frac{\omega}{a} + k^2\right)U^2 + \frac{c}{a(m+1)}U^{2m+2} = 0 \tag{7}$$

assuming that $U', U \to 0$ as $|\xi| \to \infty$.

Assuming $V = U^2$ the following ordinary differential equation is obtained:

$$(V')^2 - \alpha^2V^2 + \beta^2V^{m+2} = 0 \tag{8}$$

where

$$\alpha^2 = 4\left(\frac{\omega}{a} + k^2\right)$$

and

$$\beta^2 = \frac{4c}{a(m+1)}$$

. We can rewrite (8) as

$$V' = V\sqrt{\alpha^2 - \beta^2V^m}$$

$$\implies \frac{dV}{V\sqrt{\alpha^2 - \beta^2V^m}} = 1$$

In order to find the exact solution we integrate using MAPLE and rearrange to get

$$V(\xi) = \left[-\frac{\alpha^2}{\beta^2}\tanh^2\left(\frac{\alpha m}{2}(\xi + C)\right) - 1\right]^{\frac{1}{m}} \tag{9}$$

with C a constant of integration.

Take $U = \sqrt{V}$ to find $U(\xi)$ to be

$$U(\xi) = \left[-\frac{\alpha^2}{\beta^2}\tanh^2\left(\frac{\alpha m}{2}(\xi + C)\right) - 1\right]^{\frac{1}{2m}} \tag{10}$$

Substituting this into (2) we find the exact solution of the equation

$$u(x,t) = e^{i\theta(x,t)}\left[-\frac{\alpha^2}{\beta^2}\tanh^2\left(\frac{\alpha m}{2}(x - vt + C)\right) - 1\right]^{\frac{1}{2m}} \tag{11}$$

From (2) and (10), we have that

$$|u(x,t)| = |e^{i\theta(x,t)}||U(\xi)| = |(-1)|^{\frac{1}{2m}}\left(1 + \frac{\alpha^2}{\beta^2}\tanh^2\left(\frac{\alpha m}{2}(x + 2akt)\right)^{\frac{1}{2m}}\right) \tag{12}$$

or

$$|u(x,t)| = |U(\xi)| = \left(1 + \frac{\alpha^2}{\beta^2}\tanh^2\left(\frac{\alpha m}{2}(x + 2akt)\right)^{\frac{1}{2m}}\right) \tag{13}$$

Denoting by $A = |U|$, the amplitude of the wave, we have that

$$A = \left(1 + \frac{(\omega + ak^2)(m+1)}{c}\tanh^2\left(\sqrt{\frac{c}{a(m+1)}}m(x + 2akt)\right)\right)^{\frac{1}{2m}} \tag{14}$$

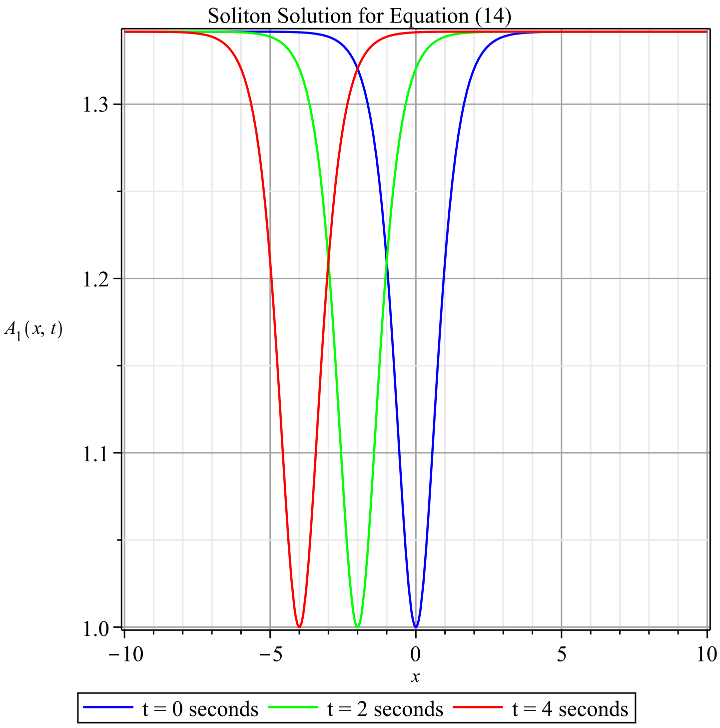

Figure 3: Graph representation for the amplitude of the soliton (equation 14) for $\omega = 0.3, a = 0.5, c = 1, k = 1, m = 1$

.

3 Simple Equation Method

In this chapter we will study the case of the NLSE where $a = 1$, $c = \mu$ and $m = 1$:

$$iu_t + u_{xx} + \mu u|u|^2 = 0 \tag{15}$$

where $u = u(x,t)$ is a complex-valued solution to the equation and μ is a constant. We will use the transformation

$$u(x,t) = e^{i(\alpha x+\beta t)}U(\xi) \quad \xi = ik(x - 2\alpha t) \tag{16}$$

where α, β and k are real constants to be determined and $U = U(\xi)$ is a complex function in ξ, in order to change the nonlinear PDE into an ODE. After subbing (16) into (15), we obtain this ODE

$$-k^2U'' - (\beta + \alpha^2)U + \mu U^3 = 0 \tag{17}$$

.

To solve this ODE using the simple equation method we write $U(\xi)$ as a finite series in terms of $\phi(\xi)$

$$U(\xi) = \sum_{i=0}^{N} a_i\phi(\xi)^i, \quad a_i \neq 0 \tag{18}$$

where a_i are independent of ξ that will be determined and $\phi = \phi(\xi)$ solves certain ordinary differential functions. In this method, this finite series is called the simplest method. We use equations where the general solution can been found easily. In this example we will use the Bernoulli equation

$$\phi'(\xi) = A\phi(\xi) + B\phi(\xi)^2. \tag{19}$$

This equation has the well known solution:

$$\phi(\xi) = \frac{Ae^{A(\xi+\xi_0)}}{1 - Be^{A(\xi+\xi_0)}} \tag{20}$$

for $A > 0, B < 0$ and where ξ_0 is a constant of integration.

To find N we balance the linear terms of highest order, U'' with the highest order of nonlinear term, U^3 of (17) to get $N + 2 = 3N \implies N = 1$ so the series in $\phi(\xi)$ becomes

$$U(\xi) = a_0 + a_1\phi(\xi). \tag{21}$$

Substituting (21) into (17) with (19) and setting each coefficient of the same power of $\phi(\xi)$ to zero, we obtain algebraic equations in $k, a_0, a_1.A, B, \alpha, \beta$ and μ:

$$\begin{aligned} -(\beta + \alpha^2)a_0 + \mu a_0^3 &= 0, \\ -k^2a_1A^2 - (\beta + \alpha^2)a_1 + 3\mu a_0^2a_1 &= 0 \\ -3k^2a_1AB + 3\mu a_0a_1^2 &= 0 \\ -2k^2a_1B^2 + \mu a_1^3 &= 0 \end{aligned} \tag{22}$$

The solve command can be used in MAPLE to get a_0, β, B:

$$a_0 = \pm\frac{kA}{2}\sqrt{\frac{2}{\mu}}, \beta = \frac{k^2A^2}{2} - \alpha^2, B = \pm\frac{a_1}{2k}\sqrt{2\mu}.$$

Using (21) and (20) and assuming $A > 0$ and $B < 0$ and substituting these equations into (17), we obtain:

$$U(\xi) = \pm\left(\frac{kA}{\sqrt{2\mu}} + a_1\left[\frac{2kAe^{A(\xi+\xi_0)}}{2k - a_1\sqrt{2\mu}e^{A(\xi+\xi_0)}}\right]\right) \tag{23}$$

The travelling solution would be

$$u(x,t) = \pm\left(\frac{kA}{\sqrt{2\mu}} + a_1\left[\frac{2kAe^{A(ik(x-2\alpha t)+\xi_0)}}{2k - a_1\sqrt{2\mu}e^{A(ik(x-2\alpha t)+\xi_0)}}\right]\right) e^{i\left(\alpha x + \left(\frac{k^2A^2}{2} - \alpha^2\right)t\right)} \tag{24}$$

Setting $B = -1$ we obtain an ordinary differential equation

$$\phi'(\xi) = A\phi(\xi) - \phi(\xi)^2 \tag{25}$$

This has a solution

$$\phi(\xi) = \frac{A}{2}\left[1 - \tanh\left(\frac{A}{2}(\xi+\xi_0)\right)\right] \tag{26}$$

where $A > 0$

Substituting (21) and (25) into (15) and setting each coefficient of the same power of $\phi(\xi)$ to zero, we obtain another set of algebraic equations

$$\begin{aligned} -(\beta+\alpha^2)a_0 + \mu a_0^3 &= 0, \\ -k^2a_1A^2 - (\beta+\alpha^2)a_1 + 3\mu a_0^2 a_1 &= 0 \\ 3k^2a_1A + 3\mu a_0 a_1^2 &= 0 \\ -2k^2a_1 + \mu a_1^3 &= 0 \end{aligned} \tag{27}$$

that can be solved in MAPLE using the solve command:

$$a_0 = \pm\sqrt{\frac{\beta+\alpha^2}{\mu}}, a_1 = \pm\sqrt{\frac{2}{\mu}}, A = -\frac{1}{k}\sqrt{2(\beta+\alpha^2)}$$

where β, α and k are arbitrary constants.

Substituting these into our solution for $U(\xi)$, (21) and using our solution for $\phi(\xi)$ we obtain:

$$U(\xi) = \sqrt{\frac{\beta+\alpha^2}{\mu}}\tanh\left(\sqrt{\frac{\beta+\alpha^2}{2k^2}}(\xi+\xi_0)\right) \tag{28}$$

This leads to the travelling wave solution

$$u(x,t) = \sqrt{\frac{\beta+\alpha^2}{\mu}}\tanh\left(\sqrt{\frac{\beta+\alpha^2}{2k^2}}(ik(x-2\alpha t)+\xi_0)\right) e^{i(\alpha x + \frac{k^2A^2}{2} - \alpha^2)t} \tag{29}$$

Will included some plots here hopefully

4 Conclusions

In this paper, exact soliton solutions of the nonlinear Schrödinger equation were found using two techniques, direct integration and the simple equation method. The first solution was found by writing the original equation as an ordinary differential equation assuming the solution is a travelling wave. This ODE was then solved using integration techniques obtaining a soliton solution. This solution was plotted in MAPLE and it was shown than the soliton's shape and speed remained unchanged for $t > 0$. To get the second soliton solution, the simplest equation method was used. To solve for $u(x, t)$, it was written in terms of a finite series of functions of ξ that solve a Bernoulli equation. I was unable to plot a soliton solution for this method using MAPLE. This can be looked at further in the future. These soliton solutions may be important in describing practical physical problems. Both of these methods may also be applied to other nonlinear partial differential equations. Techniques in MAPLE such as plotting, solving ODEs, and other equations were practiced and used frequently in this project aswell as LaTeX skills that will be used in writing future mathematical papers.

Bibliography

[1] Zakharov, V. E. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics 1972 9:2* **1968**, *9*, 190–194.

[2] Galati, L.; Zheng, S. Nonlinear Schrödinger equations for Bose-Einstein condensates. 2013; pp 50–64.

[3] Felice, D.; Barletti, L.; Gandolfi, A. A Study of a Nonlinear Schrödinger Equation for Optical Fibers CICLO XXIII, SETTORE DISCIPLINARE MAT/07 FISICA MATEMATICA. 2016; `https://arxiv.org/abs/1612.00358`.

[4] Ma, W.-X.; Chen, M. Direct search for exact solutions to the nonlinear Schrödinger equation. **2011**,

[5] Zabusky, N. J.; Kruskal, M. D. INTERACTION OF ”SOLITONS” IN A COLLISIONLESS PLASMA AND THE RECURRENCE OF INITIAL STATES. 1965.

[6] Russell, J. S. Report on Waves: Made to the Meetings of the British Association in 1842-43. 1834; `https://books.google.ie/books?hl=en&lr=&id=994EAAAAYAAJ&oi=fnd&pg=PP4&dq=Russell+JS.+Report+on+waves:+made+to+the+meetings+of+the+British+association+in+1842-43.&ots=l8n1NgKY0k&sig=pbllh6v3sWx0Oz3a8cjnFaHxHXg&redir_esc=y#v=onepage&q&f=false`.

[7] Smith, D. R.; Padilla, W. J.; Vier, D. C.; Shelby, R.; Nemat-Nasser, S. C.; Kroll, N.; Schultz, S. Left-Handed Metamaterials. *Photonic Crystals and Light Localization in the 21st Century* **2001**, 351–371.

[8] Wazwaz, A. M. A sine-cosine method for handlingnonlinear wave equations. *Mathematical and Computer Modelling* **2004**, *40*, 499–508.

[9] Wang, M.; Li, X.; Zhang, J. The (frac(G, G))-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Physics Letters, Section A: General, Atomic and Solid State Physics* **2008**, *372*, 417–423.

[10] Abdelrahman, M. A.; Zahran, E. H. M.; Khater, M. M. Exact Traveling Wave Solutions for Power Law and Kerr Law N on Linearity Using the Exp(-(ξ))-expansion Method — Request PDF. 2014; `https://www.researchgate.net/publication/276409336_Exact_Traveling_Wave_Solutions_for_Power_Law_and_Kerr_Law_N_on_Linearity_Using_the_Exp-phx-expansion_Method`.

[11] Taghizadeh, N.; Mirzazadeh, M. The simplest equation method to study perturbed nonlinear Schrödinger's equation with Kerr law nonlinearity. *Communications in Nonlinear Science and Numerical Simulation* **2012**, *17*, 1493–1499.